Journal of Global Optimization **6**: 253–267, 1995.

Solution of Linear Complementarity Problems Using Minimization with Simple Bounds*

ANA FRIEDLANDER, JOSÉ MARIO MARTÍNEZ and
SANDRA AUGUSTA SANTOS
Department of Applied Mathematics, State University of Campinas, CP 6065, 13081 Campinas SP, Brazil (Email: martinez@ccvax.unicamp.ansp.br)

(Received: 23 November 1993; accepted: 14 October 1994)

Abstract. We define a minimization problem with simple bounds associated to the horizontal linear complementarity problem (HLCP). When the HLCP is solvable, its solutions are the global minimizers of the associated problem. When the HLCP is feasible, we are able to prove a number of properties of the stationary points of the associated problem. In many cases, the stationary points are solutions of the HLCP. The theoretical results allow us to conjecture that local methods for box constrained optimization applied to the associated problem are efficient tools for solving linear complementarity problems. Numerical experiments seem to confirm this conjecture.

Key words: Horizontal linear complementarity problem, linear complementarity problem, bound constrained minimization, optimality conditions, stationary points, global minimizers.

AMS (MOS) Subject Classifications. 49M15, 65K05, 90C33

1. Introduction

We consider the horizontal linear complementarity problem (HLCP):
given $Q, R \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, find $x, z \in \mathbb{R}^n$ such that

$$Qx + Rz = b, \;\; x^T z = 0, \;\; x, z \geq 0. \tag{1}$$

This problem is a generalization of the classical linear complementarity problem (LCP):

$$z = Mx + q, \;\; x^T z = 0, \;\; x, \; z \geq 0, \tag{2}$$

where $M \in \mathbb{R}^{n \times n}, q \in \mathbb{R}^n$.

The linear complementarity problem has been studied by many authors (see [12], [3] and references therein). The reduction of (1) to (2) is trivial if R is nonsingular (take $M = -R^{-1}Q, q = R^{-1}b$). If there exists a nonsingular matrix $\tilde{R} = (v_1, \ldots, v_n)$ such that v_j is either the j-th column of Q or the j-th

* This work was supported by FAPESP (grants 90-3724-6 and 91-2441-3), CNPq and FAEP (UNICAMP).

column of R, we say that the HLCP is *reducible*. In this case, after a reordering of the variables, the HLCP takes the form

$$\tilde{Q}\tilde{x} + \tilde{R}\tilde{z} = b, \quad \tilde{x}^T\tilde{z} = 0, \quad \tilde{x}, \tilde{z} \geq 0, \tag{3}$$

which can be obviously reduced to (2).

If Q and R are such that $u^T v \geq 0$ whenever $Qu + Rv = 0$, we say that the HLCP is *monotone*. It can be proved (see [1] and [7]) that a monotone HLCP is necessarily reducible. So, for a monotone HLCP, rank $(Q, R) = n$. The first order optimality conditions of quadratic programming can be represented by an HLCP, which is monotone if the problem is convex (see [1]).

Many methods for solving linear complementarity problems can be found in the literature. However, to reduce the HLCP to (2) (when this is possible) may not be a good strategy in some situations (for example, if R and Q are sparse but $R^{-1}Q$ is dense). Therefore, methods for solving the HLCP that do not modify the structure of Q and R should be studied.

When there exists (x, z) satisfying (1) we say that the HLCP is *solvable*. If there exists $x \geq 0$ and $z \geq 0$ such that $Qx + Rz = b$, we say that the problem is *feasible*. When the HLCP is solvable, its solutions are the global minimizers of the following optimization problem:

$$\text{Minimize } \rho\|Qx + Rz - b\|^2 + x^T z \quad \text{subject to } \; x \geq 0, \; z \geq 0, \tag{4}$$

where $\rho > 0$ is an arbitrary constant. However, to find global minimizers of (4) is not easy, since stationary points may exist that are not global minimizers, even if the HLCP is monotone. For example, consider the trivial monotone horizontal linear complementarity problem (where $n = 1, Q = 0, R = 1, b = 1$):

$$z = 1, \; xz = 0, \; x \geq 0, \; z \geq 0. \tag{5}$$

The obvious solution of (5) is $x = 0$, $z = 1$. The associated (quadratic) global optimization problem is:

$$\text{Minimize } \rho(1 - z)^2 + xz \quad \text{subject to } \; x \geq 0, \; z \geq 0.$$

This problem has a stationary point at $x = 2\rho$, $z = 0$, which is not a solution of (5).

In this paper, we propose to use as auxiliary optimization problem, instead of (4), the following one:

$$\text{Minimize } \rho\|Qx + Rz - b\|^2 + (x^T z)^p \quad \text{subject to } \; x \geq 0, \; z \geq 0, \tag{6}$$

where $\rho > 0$ and $p > 1$ are arbitrary. Problem (6) was inspired by previous work of the authors on the resolution of large-scale linearly constrained optimization problems. See [5]. We will see that under general conditions (in particular, when the HLCP is monotone), the stationary points of (6) are global minimizers and, thus, solutions of (1). This result allows us to use efficient algorithms for minimization with simple bounds, which are guaranteed to converge only to stationary points. See [2], [4].

We would like to emphasize that efficient algorithms for bound constrained optimization are available. A common feature to many of these algorithms is that no factorization of matrices is needed, so that very large problems can be solved. The reduction of the LCP to an optimization problem with linear constraints, where stationary points are necessarily global minimizers under fairly general conditions, is possible (see [3] (Chapter 3, Thm. 3.5.4), [13]) but the explicit ocurrence of linear constraints limits the size of the problems that can be solved in this case.

In Section 2 of this paper we prove the main results concerning problem (6). In Section 3 we explain how to solve (1) using (6), and we show numerical experiments. Conclusions are given in Section 4.

2. Main Results

The first-order optimality (Karush–Kuhn–Tucker) conditions of (6) are:

$$2\rho Q^T(Qx + Rz - b) + p(x^T z)^{p-1} z - \gamma = 0, \tag{7}$$

$$2\rho R^T(Qx + Rz - b) + p(x^T z)^{p-1} x - \mu = 0, \tag{8}$$

$$x^T \gamma = 0, \tag{9}$$

$$z^T \mu = 0, \tag{10}$$

$$x \geq 0, \quad z \geq 0, \quad \mu \geq 0, \quad \gamma \geq 0. \tag{11}$$

If (x, z) satisfies (9)–(11), with γ and μ defined by (7) and (8), we say that (x, z) is a *stationary point* of problem (6). Modern algorithms for box constrained optimization are usually successful for finding stationary points (frequently local minimizers) of minimization problems with simple bounds, like (6). So, it is important to detect situations where these stationary points are global minimizers. The following theorem gives some critical properties of stationary points of (6) that *are not* solutions of the HLCP.

THEOREM 1. *Assume that the HLCP (1) is feasible and that (x_*, z_*), a stationary point of (6), is not a solution of the HLCP. Let us define $r_* = Qx_* + Rz_* - b$. Then,*

$$r_* \neq 0, \tag{12}$$

$$x_*^T z_* > 0 \tag{13}$$

and

$$r_*^T R Q^T r_* > 0. \tag{14}$$

Proof. Let us prove first (13). We proceed by contradiction. If $x_*^T z_* = 0$, we have, by (7)–(11), that

$$2\rho Q^T(Qx_* + Rz_* - b) - \gamma = 0, \tag{15}$$

$$2\rho R^T(Qx_* + Rz_* - b) - \mu = 0, \tag{16}$$

$$x_*^T \gamma = 0 \tag{17}$$

$$z_*^T \mu = 0 \tag{18}$$

$$x_* \geq 0,\ z_* \geq 0,\ \mu \geq 0,\ \gamma \geq 0. \tag{19}$$

Now, (15)–(19) are necessary and sufficient conditions for a global minimizer of the following convex quadratic minimization problem:

$$\text{Minimize } \rho\|Qx + Rz - b\|^2 \quad \text{subject to} \quad x \geq 0,\ z \geq 0. \tag{20}$$

Since, by hypothesis, the HLCP is feasible, it turns out that (x_*, z_*) is a global solution of (20) with minimum value zero, that is,

$$Qx_* + Rz_* - b = 0.$$

Therefore, by (19) and the initial assumption, (x_*, z_*) is a solution of the HLCP, contradicting the hypothesis. So, (13) is proved.

Let us now prove (14). Again, let us suppose, by contradiction, that

$$r_*^T RQ^T r_* \leq 0.$$

Now, by (7), there exist $\gamma \geq 0$, $\mu \geq 0$ such that

$$(p(x_*^T z_*)^{p-1}x_* - \mu)^T[Q^T 2\rho(Qx_* + Rz_* - b) + p(x_*^T z_*)^{p-1}z_* - \gamma] = 0 \tag{21}$$

By (8), we have that

$$p(x_*^T z_*)^{p-1}x_* - \mu = -2\rho R^T(Qx_* + Rz_* - b). \tag{22}$$

But, if $r_*^T RQ^T r_* \leq 0$, (21) and (22) imply that

$$[p(x_*^T z_*)^{p-1}x_* - \mu]^T[p(x_*^T z_*)^{p-1}z_* - \gamma] \leq 0. \tag{23}$$

Therefore,

$$p^2(x_*^T z_*)^{2p-1} - p(x_*^T z_*)^{p-1}(x_*^T\gamma) - p(\mu^T z_*)(x_*^T z_*)^{p-1} + \mu^T\gamma \leq 0. \tag{24}$$

Then, by (24), (9) and (10), we have that

$$p^2(x_*^T z_*)^{2p-1} + \mu^T\gamma \leq 0.$$

So, by (11), the complementarity equation $x_*^T z_* = 0$ holds. This is a contradiction, since in the first part of the theorem we proved (13). Therefore, (14) is also proved.

Finally, observe that (14) trivially implies (12). QED

The following corollaries state some straightforward consequences of Theorem 1.

COROLLARY 1. *Assume that (x_*, z_*) is a stationary point of (6), and define $r_* = Qx_* + Rz_* - b$. Then, the following propositions hold:*

(a) If $r_* = 0$, then (x_*, z_*) is a solution of the HLCP.

(b) If the HLCP is feasible and $x_*^T z_* = 0$, then (x_*, z_*) is a solution of the HLCP.

(c) If the HLCP is feasible and

$$r_*^T R Q^T r_* \leq 0,$$

then (x_*, z_*) is a solution of the HLCP.

(d) If the HLCP is feasible and RQ^T is negative semidefinite, then (x_*, z_*) is a solution of the HLCP.

COROLLARY 2. *Assume the hypotheses of Corollary 1. Let $\tilde{R} = (v_1, \ldots, v_n)$ be such that v_j is either the j-th column of Q or the j-th column of R, and $\tilde{Q} = (w_1, \ldots, w_n)$ such that w_j is the j-th column of R when v_j is the j-th column of Q and vice-versa. Then,*

(a) If the HLCP is feasible and $r_*^T \tilde{R} \tilde{Q}^T r_* \leq 0$, then (x_*, z_*) is a solution of the HLCP.

(b) If the HLCP is feasible and $\tilde{R}\tilde{Q}^T$ is negative semidefinite, then (x_*, z_*) is a solution of HLCP. (In particular, if QR^T is negative semidefinite, then (x_*, z_*) is a solution of the HLCP.)

In the following lemma we give a characterization of monotone horizontal linear complementarity problems that will allow us to prove that stationary points of (6) are global solutions of feasible monotone problems.

LEMMA 1. *Suppose that, after a possible renaming of the variables, an HLCP can be expressed as*

$$Qx + Rz = b, \quad x^T z = 0, \quad x, z \geq 0, \tag{25}$$

where R is nonsingular and RQ^T is negative semidefinite. Then, the HLCP is monotone. Reciprocally, if the HLCP is monotone, then, for all possible renaming of the variables with R nonsingular, RQ^T must be negative semidefinite.

Proof. Suppose that the HLCP can be written in the form (25) with R nonsingular. Let u, v be such that

$$Qu + Rv = 0.$$

Then $v = -R^{-1}Qu$. We define $w = R^{-T}u$. Since RQ^T is negative semidefinite we have that

$$\begin{aligned} 0 \le -w^T RQ^T w &= -w^T RQ^T R^{-T} R^T w \\ &= -u^T Q^T R^{-T} u = -u^T R^{-1} Q u = u^T v. \end{aligned}$$

Therefore, the HLCP is monotone. Reciprocally, assume that the HLCP is monotone and let $w \in I\!R^n$ be arbitrary. We define $u = R^T w, v = -R^{-1}Qu$. So, $Qu + Rv = 0$. Since the problem is monotone, we have that

$$\begin{aligned} 0 \le u^T v &= -u^T R^{-1} Q u = -u^T Q^T R^{-T} u \\ &= -w^T RQ^T R^{-T} R^T w = -w^T RQ^T w. \end{aligned}$$

Therefore, RQ^T is negative semidefinite. QED

COROLLARY 3. *Assume that the HLCP is monotone and feasible, and that* (x_*, z_*) *is a stationary point of (6). Then* (x_*, z_*) *is a solution of the HLCP.*

Proof. As we mentioned in the introduction, if the HLCP is monotone, then, after a possible renaming of the variables, it can be expressed in the form (25). So, the conclusion follows from Lemma 1 and Corollary 2. QED

REMARKS. We showed that, if the HLCP is feasible, the condition "RQ^T negative semidefinite" is sufficient to guarantee that all the stationary points of (6) are global minimizers. This condition is more general than the monotonicity of the problem. For example, consider an HLCP with Q singular and $R = -Q$. In this case RQ^T is clearly negative semidefinite, but the problem is not monotone, since rank $(Q, R) < n$. (See [7], [1].)

Theorem 1 and its corollaries give us a strong feeling that, very likely, a good "local" method applied to (6) (that is, a method whose convergence is guaranteed to stationary points) will be effective for solving the HLCP. In fact, according to these results, a stationary point of a feasible HLCP which is not a global minimizer of (6) must satisfy $r_*^T \tilde{R}\tilde{Q}^T r_* > 0$ for all possible partitions of (R, Q). This condition is strong, so, roughly speaking, even if the problem is not monotone, the chances of the value $r_*^T \tilde{R}\tilde{Q}^T r_*$ being nonpositive for a stationary point of (6) are reasonable. We will find numerical evidence of our practical conjecture in Section 3.

3. Numerical Experiments

Problem (6) satisfies the following "global property":

The objective function is nonnegative for all points on the feasible region Ω *and it vanishes if and only if the corresponding minimizer solves the HLCP.*

In the case of (6), $\Omega = \{(x, z) \in I\!R^n \times I\!R^n \mid x, z \ge 0\}$. Some authors introduced unconstrained minimization problems that satisfy the global property above, associated to nonlinear complementarity problems and variational inequalities. See [11] and [6]. The Mangasarian–Solodov problem associated to the linear complementarity problem (2) is

$$\text{Minimize } x^T(Mx+q) + \frac{1}{2\alpha}[\|(-\alpha(Mx+q)+x)_+\|^2 - \|x\|^2 \tag{26}$$

$$+\|(-\alpha x + Mx + q)_+\|^2 - \|Mx+q\|^2],$$

where $\alpha > 1$ is arbitrary and $[v_+]_i$ =max $(0, v_i)$. It is proved in [11] that this problem satisfies the global property. However, even if M is positive semidefinite (monotone problem), local minimizers of (26) could not be solutions of (2). For example, it is easy to see that all the points $\{x \in I\!R \mid x > \alpha\}$ are local minimizers of the Mangasarian-Solodov problem associated to (5). Kanzow [10] proved that, when M is positive definite and the LCP is solvable, stationary points of (26) are solutions of the LCP. Similar properties hold for Fukushima's function [6]. Observe that, in general, the objective function of (26) has continuous first derivatives but discontinuous second derivatives.

Both the Mangasarian-Solodov approach and our approach (6) (with $Q = M$, $R = -I$ and $b = -q$) reduce the LCP to a minimization problem where the feasible set is simple ($I\!R^n$ for (26) and the positive orthant of $I\!R^{2n}$ in the case of (6)). Therefore, in both cases, we can use algorithms that do not use factorization of matrices, penalization, or estimation of Lagrange multipliers for solving the optimization problem.

In this research, we employed the method for minimization of general functions with bound constraints described in [4] and [5]. This method produces a sequence of approximations (x_k, z_k) such that, for all $k = 0, 1, 2, \ldots$, (x_{k+1}, z_{k+1}) is an approximate critical point of a quadratic on a $2n$-dimensional box. (In the case of (26) the iterates are $\{x_k\} \subset I\!R^n$.) It has been proved in [4] that limit points of the sequence generated by the algorithm are stationary points of the minimization problem. For finding the approximate critical point of each quadratic we use an algorithm that combines conjugate gradients and gradient projection techniques. Since no manipulation of matrices is present, the method is able to deal with large problems. The parameters used for running this method were the ones recommended in [5], including the tolerances used to declare "convergence". In addition to the usual stopping criteria of the box constrained minimization algorithm, we also stopped the execution when

$$f(x_k, z_k) \leq 10^{-8}$$

where f is the objective function of problem (6). A similar convergence criterion was used in connection to (26). The maximum allowed number of iterations was 500. However, this maximum was reached only in the third set of experiments (Problem VD2 with $a \neq 0$).

For all our experiments, we used a PC-486 type computer, DOS operating system, Microsoft FORTRAN and double precision. We performed four sets of numerical experiments.

3.1. First set of experiments

We generated a set of linear complementarity problems as follows: the matrix $M \in I\!R^{n\times n}, n = 10$, was generated as

$$M = S + A \tag{27}$$

where S is symmetric and A is skew-symmetric. Clearly, M is positive semidefinite if and only if S is. We defined $(\pi, \theta, \nu) \equiv (\pi(S), \theta(S), \nu(S))$ the number of positive, null and negative eigenvalues of S. The matrix S of each problem was generated as

$$S = (I - 2\frac{v_n v_n^T}{v_n^T v_n}) \cdots (I - 2\frac{v_1 v_1^T}{v_1^T v_1}) D (I - 2\frac{v_1 v_1^T}{v_1^T v_1}) \cdots (I - 2\frac{v_n v_n^T}{v_n^T v_n}) \tag{28}$$

where the vectors $v_i \in I\!R^n$ were randomly generated with entries between 0 and 1, and D was a diagonal matrix with π entries randomly generated between 0 and 10, ν entries randomly generated between 0 and -10 and θ null entries. The matrix A was generated with random entries between -10 and 10 on its strict upper triangular elements a_{ij}, with $a_{ji} = -a_{ij}$ for all i, j. The solution x_* of the LCP was generated taking $[x_*]_i = 0$ with probability $\frac{1}{2}$, and $[x_*]_i$ chosen at random between 0 and 10 for the positive entries. After the computation of x_* and M, we computed $q \in I\!R^n$ in the following way:

For $i = 1, \dots, n$, if $[x_*]_i > 0$, we set $q_i = -[Mx_*]_i$, while, if $[x_*]_i = 0$, we set $q_i = -[Mx_*]_i +$ [a random number between 0 and 10].

We tried to solve the LCP's defined above using the auxiliary subproblem (6) with $p = 2, \rho = 1$. We used the algorithm for box-constrained minimization introduced in [4] and [5] where the initial approximation (x_0, z_0) was generated with random entries between 0 and 10. In addition, we tried to solve the same problems using the unconstrained minimization problem (26) ($\alpha = 2$) and x_0 as initial approximation. Since the LCP's generated are solvable, their set of solutions, the set of global minimizers of (6) and the set of global minimizers of (26) are identical. For each triplet (π, θ, ν) we generated ten different problems. In Table 1 we report the number of cases in which the algorithm converged to the solution of the LCP using (6) and (26) respectively. In the parentheses we indicate the average number of iterations of the successful runs.

In Table I we observe that the efficiency of the approach (6) for finding solutions of the LCP is between 60% (when π is small and ν is large) and 100% (when $\nu = 0$). On the other hand, the approach based on (26) is very good when π

TABLE I. Random problems varying π, θ, ν.

π	θ	ν	Problem (6)	Problem (26)
0	2	8	7 (14)	1 (6)
1	2	7	6 (14)	2 (8)
2	2	6	8 (11)	4 (12)
3	2	5	8 (13)	7 (46)
4	2	4	9 (16)	7 (31)
5	2	3	10 (18)	6 (41)
6	2	2	9 (18)	9 (33)
7	2	1	9 (16)	10 (37)
8	2	0	10 (14)	10 (44)
0	4	6	6 (14)	1 (5)
3	4	3	7 (50)	4 (24)
6	4	0	10 (16)	10 (31)
0	7	3	7 (16)	2 (21)
3	7	0	10 (17)	10 (46)
0	10	0	10 (15)	0

is large but fails frequently when π is small.

3.2. Second Set of Experiments

We generated non-reducible horizontal linear complementarity problems. We computed $\overline{Q} \in I\!R^{(n-1)\times n}$ with random entries between -10 and 10 and $\overline{R} = -\overline{Q} + \beta T$, where $T \in I\!R^{(n-1)\times n}$ had random entries between -10 and 10, and $\beta \in [0, 10]$. (As before, we used $n = 10$.) Then, we defined

$$Q^T = (\overline{Q}^T, \overline{Q}^T \underline{e}/(n-1))$$

and

$$R^T = (\overline{R}^T, \overline{R}^T \underline{e}/(n-1)),$$

where $\underline{e} = (1, \ldots, 1)^T \in I\!R^{n-1}$. So, the rank of (Q, R) is at most equal to $n - 1$ and, consequently, the problem is not reducible. If $\beta = 0$, we have $R = -Q$ so that the condition "RQ^T negative semidefinite" is satisfied. As β is incremented, this hypothesis becomes less probable. The coordinate j of x_* was zero with probability $\frac{1}{2}$ and a random number between 0 and 10 with the same probability. When this coordinate is not zero, the corresponding coordinate of z_* is zero. Otherwise, the coordinate j of z_* is a random number between 0 and 10. Finally, we computed $b = Qx_* + Rz_*$. The initial point (x_0, z_0) for running the box constrained minimizer with problem (6) ($\rho = 1, p = 2$) was generated as in the first set of experiments. We studied the influence of β on the efficiency of the approach (6) for

TABLE II. Non-reducible problems

β	Successful cases
0.	10 (8)
1.	10 (9)
2.	9 (9)
3.	9 (11)
4.	10 (12)
5.	10 (12)
6.	10 (12)
7.	10 (15)
8.	10 (17)
9.	10 (16)
10.	10 (18)

finding global solutions of (6), keeping in mind that the coincidence of stationary points and global minimizers is only guaranteed for $\beta = 0$. For each value of β we ran ten problems. In Table II we report the number of cases where the global minimizer was found and, between parentheses, the average number of iterations of the successful runs. In this table we also observe that the solution of the HLCP using (6) was found almost always, even when we do not know if the hypothesis "RQ^T positive semidefinite" holds.

3.3. Third Set of Experiments

We solved two classical variably dimensioned linear complementarity problems using (6) ($\rho = 1, p = 2$). For these problems, we report the number of iterations (ITER), the number of functional evaluations (FE) and the execution time, in seconds (TIME) used by the box minimizer.

In all these tests we used as initial point (x_0, z_0), where

$$x_0 = (a, \ldots, a)^T,$$

$$[z_0]_i = \max\{0,\ [Mx_0 + q]_i\},\ i = 1, \ldots, n,$$

and $a \geq 0$ is a given parameter.

The tests problems were the following:

PROBLEM VD1. See [8], [12] (chapter 6) and [9].

$$n \text{ variable}, \quad q = (-1, \ldots, -1)^T,$$

$$[M]_{ii} = 1,\ i = 1, \ldots, n,$$

$$[M]_{ij} = 2,\ i = 1, \ldots, n-1,\ j > i,$$

$$[M]_{ij} = 0,\ i = 1, \ldots, n-1,\ j < i.$$

PROBLEM VD2. See [8], [12] (chapter 6) and [9].

$$n \text{ variable}, \quad q = (-1, \ldots, -1)^T,$$

$$[M]_{ii} = 4(i-1) + 1,\ i = 1, \ldots, n,$$

$$[M]_{ij} = [M]_{ii} + 1,\ i = 1, \ldots, n-1,\ j = i+1, \ldots, n,$$

$$[M]_{ij} = [M]_{jj} + 1\ j = 1, \ldots n-1,\ i = j+1, \ldots, n.$$

The results are given in Table III.

The experiments in Table III seem to confirm that using (6) and a box-constrained optimization solver is a reliable approach for solving many LCP's, even when the problem is not monotone. (The matrix M of problem VD2 is positive semidefinite, so the problem is monotone, but this is not the case of Problem VD1.) In the case of Problem VD1, Murty [12] reports that the number of iterations used by both Lemke's algorithm and the principal pivoting method of Cottle and Dantzig grows exponentially with n. The same seems to happen with the method of Harker and Pang [8]. Kanzow [9] shows that in his Newton-like method the number of iterations grows linearly with n. We observe that, using our approach, the number of iterations grows very slowly with n. In Problem VD2, exponential behavior also occurs for Lemke's algorithm and the principal pivoting method (see [12]). On the other hand, the number of iterations used by our method remains approximately constant for $n < 200$ and grows slowly as a function of n for $n \geq 200$. It is worth mentioning that the performance of Problem VD1 is independent of the initial point. For Problem VD2, however, the convergence becomes slower when $a \neq 0$, to the extent of stopping by achieving the maximum allowed number of iterations.

Observe that the matrix M of Problem VD2 is dense. So, it is remarkable that we could solve this problem for $n = 1000$ with a modest computer using only 3169 seconds of CPU time. This performance should be completely impossible if algorithms using factorization of matrices were employed.

3.4. Fourth set of experiments

We also used our method for solving the problem given by Murty ([12], p. 354), where we chose the vector q as

$$q = (1, \ldots, 1, -n+2, n-2)^T.$$

TABLE III. Variably dimensioned linear complementarity problems

Problem	n	a	ITER	FE	TIME	Final f
VD1	10	0	2	3	0.17	2.E-29
	20	0	3	4	0.33	7.E-25
	30	0	3	4	0.39	7.E-26
	40	0	3	4	0.55	9.E-25
	50	0	3	4	0.71	4.E-28
	60	0	3	4	1.05	5.E-22
	70	0	3	4	1.21	6.E-26
	80	0	3	4	1.59	1.E-23
	90	0	4	5	2.31	5.E-22
	100	0	4	5	2.53	5.E-22
	110	0	5	6	3.79	4.E-22
	120	0	5	6	4.67	3.E-22
	130	0	4	5	4.34	2.E-23
	140	0	5	6	5.71	1.E-25
	150	0	5	6	7.31	3.E-23
	200	0	6	7	13.40	8.E-23
	250	0	6	7	21.48	6.E-23
	300	0	6	7	29.87	4.E-19
	350	0	6	7	41.85	9.E-20
	400	0	5	6	49.88	8.E-20
	450	0	5	6	63.71	4.E-19
	500	0	5	6	78.87	8.E-19
	550	0	8	9	123.09	1.E-23
	600	0	8	9	135.99	2.E-18
	650	0	7	8	163.79	4.E-18
	700	0	5	6	170.26	5.E-19
	750	0	7	8	228.76	1.E-17
	800	0	6	7	233.27	6.E-22
	850	0	5	6	255.46	3.E-17
	900	0	7	8	327.02	2.E-17
	950	0	6	7	342.35	2.E-17
	1000	0	7	8	403.87	5.E-19
VD2	10	0	4	5	0.22	2.E-27
	20	0	4	5	0.44	7.E-27
	30	0	4	5	0.77	2.E-24
	40	0	4	5	1.32	4.E-23
	50	0	5	6	1.92	3.E-23
	60	0	5	6	3.07	3.E-24
	70	0	5	6	3.90	2.E-23
	80	0	6	7	5.77	2.E-22
	90	0	7	8	7.63	7.E-22

TABLE III. Continued

Problem	n	a	ITER	FE	TIME	Final f
	100	0	6	7	9.50	2.E-21
	110	0	6	7	12.42	5.E-22
	120	0	7	8	14.66	6.E-22
	130	0	8	9	19.78	1.E-21
	140	0	7	8	21.09	5.E-23
	200	0	9	10	57.06	2.E-22
	300	0	8	9	135.94	2.E-21
	400	0	11	12	315.61	2.E-22
	500	0	11	12	500.92	5.E-20
	600	0	13	14	842.44	1.E-12
	700	0	13	14	1275.59	3.E-12
	800	0	15	16	1760.86	6.E-12
	900	0	19	20	2367.18	2.E-12
	1000	0	25	26	3168.60	3.E-16

This should be a very hard example for our approach, because the matrix M is negative definite. So, we did not expect a very good performance of our procedure in this case. In fact, in some cases, with $\rho = 1$ and $p = 2$ we could only obtain stationary points of (6) which were not global minimizers. Thus, we decided to carry out different experiments with this problem, increasing the value of ρ. We expected that better results could be obtained giving larger weights to the difficult constraint $Mx + q - z = 0$. The results are in Table IV. It is apparent from these numerical results that our approach looks promising, even in those situations where there exist stationary points such that $r_*^T RQ^T r_* > 0$. Observe that in most cases we found values of ρ which led us to global minimizers of (6). In a few cases ($n = 700, 800$ and 850) the value of the objective function at the final point is small, but it is greater than 10^{-8}. However, even in these cases we could verify that the final point was an approximate global minimizer.

4. Final Remarks

The reduction of a nonlinear programming problem to a minimization problem with simple bounds is very attractive, considering the present state of art of optimization software development, because efficient large scale optimization solvers with box constraints exist. These algorithms are only *local*, in the sense that convergence is guaranteed only to first order stationary points of the minimization problem. The most natural bound constrained minimization problems associated to (1) is the quadratic programming problem (4). However, even when the HLCP is monotone and solvable, stationary points of (4) can exist that are not global minimizers. We proved that this is not the situation of problem (6), for which stationary points are

TABLE IV. Murty's negative definite problem

Problem	n	a	ρ	ITER	FE	TIME	Final f
Murty	50	0.8	1	3	41	1.04	57.3
			10	2	38	0.77	7270
			20	16	57	1.92	1.E-9
	100		1	13	34	3.46	3.E-12
	150		1	18	110	5.87	5710
			10	22	82	7.85	4.E-9
	200		1	20	29	5.44	7.E-9
	250		1	12	37	6.92	9.E-9
	300		1	9	19	5.10	1.E-15
	350		1	3	41	4.34	3030
			10	3	41	4.11	30300
			20	5	8	1.81	5.E-17
	400		1	3	41	5.22	39600
			10	3	41	5.77	396000
			60	24	87	20.59	3.E-10
	450		1	3	41	5.82	50200
			10	3	41	6.86	502000
			30	10	17	7.58	5.E-10
	500		1	3	41	6.32	62000
			10	3	41	7.09	620000
			1200	7	21	6.31	1.E-10
	550		1	3	41	6.65	75100
			10	3	41	6.59	751000
			1400	8	22	7.47	9.E-12
	600		1	3	41	7.69	89400
			10	3	41	9.06	894000
			80	13	25	10.27	3.E-12
	650	0.8	1	3	41	8.35	105000
			10	3	41	9.28	1050000
			40	8	15	7.31	4.E-22
	700		1	3	41	9.22	122000
			10	3	41	10.97	1220000
			800	18	98	39.71	3.E-4
	800		1	3	41	9.88	159000
			10	3	41	10.49	1590000
			50	8	35	11.97	5.E-5
	850		1	3	41	10.55	180000
			10	3	41	11.81	1800000
			1100	21	100	50.04	2.E-3
	900		1	3	41	11.15	202000
			10	3	41	12.41	2020000
			50	13	28	16.64	2.E-16
	950		1	3	41	11.26	225000
			10	3	41	10.88	2250000
			50	13	31	18.29	3.E-9
	1000		1	3	41	12.41	249000
			10	3	41	14.77	2490000
			20000	12	37	22.85	8.E-18

necessarily global minimizers under conditions that are strictly more general than the monotonicity of the HLCP.

Current research involves the extension of the approach presented in this paper to nonlinear complementarity problems and variational inequalities.

References

1. J.F. Bonnans and C.C. Gonzaga, "Convergence of interior point algorithms for the monotone linear complementarity problem", Technical Report, INRIA, 1993.
2. A. R. Conn, N. I. M. Gould and Ph. L. Toint, "Global convergence of a class of trust region algorithms for optimization with simple bounds", *SIAM Journal on Numerical Analysis* **25** (1988) 433–460. See also *SIAM Journal on Numerical Analysis* **26** (1989) 764–767.
3. R.W. Cottle, J-S Pang and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, Boston, 1992.
4. A. Friedlander, J.M. Martínez and S.A. Santos, "A new trust-region algorithm for bound constrained minimization", *Applied Mathematics and Optimization* **30** no. 3 (1994) 235–266.
5. A. Friedlander, J.M. Martínez and S.A. Santos, "On the resolution of linearly constrained convex minimization problems", *SIAM Journal on Optimization* **4** (1994) 331–339.
6. M. Fukushima, "Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems", *Mathematical Programming* **53** (1992) 99–110.
7. O. Güler, "Generalized linear complementarity problems and interior point algorithms for their solutions" Internal Report, Department of Industrial Engineering an Operations Research, University of California, 1993.
8. P.T. Harker and J-S Pang, "A damped Newton method for the linear complementarity problem", in *Computational solution of nonlinear systems of equations*, edited by E.L. Allgower and K. Georg, Lectures in Applied Mathematics, Volume 26, AMS, Providence, Rhode Island (1990) 265–284.
9. C. Kanzow, "Global convergence properties of some iterative methods for linear complementarity problems", Preprint 72, Institut für Angewandte Mathematik der Universität Hamburg, 1993.
10. C. Kanzow, "Nonlinear complementarity as unconstrained optimization", Preprint, Institut für Angewandte Mathematik der Universität Hamburg, February 1994.
11. O.L. Mangasarian and M.V. Solodov, "Nonlinear complementarity as unconstrained and constrained minimization", *Mathematical Programming* **62** (1993) 277–297.
12. K.G. Murty, *Linear Complementarity, Linear and Nonlinear Programming*, Sigma Series in Applied Mathematics 3, Heldermann Verlag, Berlin, Germany, 1988.
13. P.M. Pardalos and J.B. Rosen, "Global optimization approach to the linear complementarity problem", *SIAM Journal on Scientific and Statistical Computing* **9** (1988) 341–353.